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Meromorphic solutions of difference Painlevé equations

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Abstract
We show that, under certain conditions, some difference Painlevé equations have nontrivial meromorphic solutions in the whole complex plane. These meromorphic solutions are obtained by analytic continuation of asymptotic solutions given in sectors of zero opening angle. The existence of these asymptotic solutions is shown by using fundamental matrix solutions of the associated linear systems with double unit characteristic roots and with coefficients expressed by asymptotic series in fractional powers.

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1. Introduction

The six Painlevé equations are characterized by the Painlevé property: that, for every solution, all the movable singularities are poles. Discrete analogues of Painlevé equations, which are called discrete Painlevé equations, were discovered in various problems in mathematical physics. For example, the non-autonomous mapping

$$y_{n+1} + y_{n-1} = \frac{(an+b)y_n + c}{1 - y_n^2},$$

which appears in connection with unitary matrix models of two-dimensional quantum gravity [14], is known as the discrete P_{II} [3, 15, 16]. The limiting procedure $y_n = z_n/\delta, an + b = -(a'n + b')/\delta^2, c = -c'/\delta^3$ $(\delta \to 0)$ yields the discrete P_{I}

$$z_{n+1} + z_{n-1} = \frac{a'n + b'}{z_n} + \frac{c'}{z_n^2},$$

which is also obtained from Bäcklund transformations for the third Painlevé equation [3]. If $a' = -1$, then the continuous limit $z_n = -\varepsilon^{-5/2} + \varepsilon^{-1/2}u(t), n = \varepsilon^{-1}t, b' = 6\varepsilon^{-5}$,

$c' = 4\varepsilon^{-15/2}$ $(\varepsilon \to 0)$ maps this equation to the first Painlevé equation $u'' = 6u^2 + t$ [3]. In general, discrete Painlevé equations admit the singularity confinement property, which has been considered to correspond to the Painlevé property and to be an effective criterion for characterizing them ([5], see also [4]).

From a general mapping of the form $y_{n+1} + y_{n-1} = R(n, y_n)$ with $R(n, y)$ rational in (n, y), we obtain the corresponding difference equation

$$y(x + 1) + y(x - 1) = R(x, y(x))$$

in the complex domain, by replacing n with the complex variable x. As an analogue of the Painlevé property for the difference equation above or for

$$y(x + 1)y(x - 1) = R(x, y(x)),$$

Ablowitz, Halburd and Herbst [1] proposed the property that it has a non-rational meromorphic solution of finite order, and proved that this condition implies $\deg_y(R) \leqslant 2$ (see also [17]). From the category of difference equations of the form

$$y(x + 1) + y(x - 1) = R^*(x, y(x)),$$

where $R^*(x, y)$ is rational in y and meromorphic in x, under the supposition that there exists an admissible meromorphic solution of finite order, Halburd and Korhonen [7] derived a class of difference equations containing

$$y(x + 1) + y(x - 1) = \frac{\alpha x + \beta}{y(x)} + \gamma, \tag{1.1}$$

$$y(x + 1) + y(x - 1) = \frac{\alpha x + \beta}{y(x)} + \frac{\gamma}{y(x)^2}, \tag{1.2}$$

$$y(x + 1) + y(x - 1) = \frac{(\alpha x + \beta)y(x) + \gamma}{1 - y(x)^2} \tag{1.3}$$

with $\alpha, \beta, \gamma \in \mathbb{C}$ (see also [6]), which may be regarded as difference Painlevé equations. Equation (1.2) (respectively, (1.3)) corresponds to the discrete P_{I} (respectively, the discrete P_{II}) mentioned above, and the discrete version of (1.1) is known as another type of discrete P_{I} [15]. Difference equations including (1.1), (1.2) and (1.3) are reviewed in [8] mainly from a view point of complex analysis and value distribution theory.

Classical special functions such as the hypergeometric function, the Bessel function and the gamma function satisfy differential or difference equations, and the Painlevé transcendents have a plenty of interesting properties as nonlinear special functions. In view of this fact, we may expect that each of these difference Painlevé equations defines new interesting special functions, if it admits nontrivial meromorphic solutions. Indeed, some of the important open problems posed in [8] are concerned with the existence of meromorphic solutions. Equation (1.2) (respectively, (1.3)) with $\alpha = 0$ has a general solution expressible in terms of the elliptic function $\wp(z)$ (respectively, $\mathrm{sn}(z)$) ([6, 8]). However, except for such special cases, the existence of meromorphic solutions has not been established. In this paper we show the existence of nontrivial meromorphic solutions of (1.1), (1.2) and (1.3) under the conditions $\gamma^2/\alpha \in \mathbb{R}$, $\gamma^2/\alpha^3 \in \mathbb{R}$ and $\alpha \in \mathbb{R}\setminus\{0\}$, respectively.

We discuss meromorphic solutions mainly for (1.1). We show that (1.1) has an asymptotic solution in a certain domain containing the positive real axis, which may be continued meromorphically to the whole complex plane. As a first step we construct formal solutions of (1.1) in section 3. In section 4, using a holomorphic function asymptotic to one of these formal solutions, we derive a nonlinear difference equation equivalent to (1.1). The

Jacobi matrix related to the linear part of this equation has double unit eigenvalues. Many of the works on nonlinear difference equations (for example, [9–12]) treated ones with no unit eigenvalues, except for some cases with a single unit eigenvalue [13, 18, 19]. These results are not applicable to our case. In proving the asymptotic validity of our solution, we use linearly independent solutions of a linear difference equation which approximates this equivalent nonlinear equation. Section 5 is devoted to finding a fundamental matrix solution of this associated linear system with double unit characteristic roots and with coefficients expressed by asymptotic series in fractional powers. The proofs of the results for (1.1) are given in section 6. The results for (1.2) and (1.3) are obtained by analogous arguments, which are illustrated in sections 7 and 8.

2. Main results

In what follows we suppose that $\alpha \neq 0$, and the branches of $x^{1/\nu}$ ($\nu \in \mathbb{N}$) are taken so that $x^{1/\nu} > 0$ for $x > 0$. For $a > 0, r > 1$, let us set

$$D^a_{+\infty}(r) := \{x \,|\, |\mathrm{Im}\, x| < |\mathrm{Re}\, x|^a, \mathrm{Re}\, x > r\}.$$

In each appearance of this symbol with no explanation on r, we suppose that the constant r is chosen sufficiently large. If $a < 1$, this may be regarded as a sector of zero opening angle with the centre line $\mathbb{R}$.

2.1. Equation (1.1)

We begin with the result on formal solutions of (1.1).

Proposition 2.1. *Equation (1.1) admits formal solutions of the form*

$$\Phi_\pm(\alpha, \beta, \gamma, x) = \pm\sqrt{\alpha/2}x^{1/2} + \sum_{j\geqslant 0} c^\pm_j x^{-j/2}, \tag{2.1}$$

where $c^\pm_j$ ($j = 0, 1, 2, \ldots$) are rational functions of $\sqrt{\alpha/2}$, β and γ, in particular

$$c^\pm_0 = \frac{\gamma}{4}, \qquad c^\pm_1 = \pm\frac{8\beta + \gamma^2}{32\sqrt{\alpha/2}}.$$

These formal solutions satisfy $\Phi_-(\alpha, \beta, -\gamma, x) = -\Phi_+(\alpha, \beta, \gamma, x)$.

Our results on asymptotic solutions of (1.1) are stated as follows:

Theorem 2.2. *Suppose that $\lambda := \gamma/\sqrt{\alpha/2} \in \mathbb{R}$.*

(1) If $\lambda < 0$, then (1.1) has a solution $\varphi_-(x) = \varphi_-(\alpha, \beta, \gamma, x)$ with the properties:

(a) $\varphi_-(x)$ is holomorphic in the domain $D^{1/4}_{+\infty}(r)$;

(b) $\varphi_-(x)$ admits the asymptotic representation

$$\varphi_-(x) \sim \Phi_-(\alpha, \beta, \gamma, x)$$

as $x \to \infty$ through $D^{1/4}_{+\infty}(r)$.

(2) If $\lambda > 0$, then (1.1) has the solution $\varphi_+(x) := -\varphi_-(\alpha, \beta, -\gamma, x)$ holomorphic in $D^{1/4}_{+\infty}(r)$.

Remark 2.1. Since $\Phi_-(\alpha, \beta, -\gamma, x) = -\Phi_+(\alpha, \beta, \gamma, x)$ (cf proposition 2.1), we have $\varphi_+(x) \sim \Phi_+(\alpha, \beta, \gamma, x)$ as $x \to \infty$ through $D^{1/4}_{+\infty}(r)$.

Under other conditions there exist solutions holomorphic in the domain

$$D^{1/4}_{-\infty}(-r) := \{-x \,|\, x \in D^{1/4}_{+\infty}(r)\} \qquad (r > 0),$$

which contains the negative real axis.

Theorem 2.3. *Suppose that* $\tilde{\lambda} := \gamma/\sqrt{-\alpha/2} \in \mathbb{R}$. *If* $\tilde{\lambda} < 0$ *(respectively,* $\tilde{\lambda} > 0$*), then (1.1) has the solution*

$$\widetilde{\varphi}_-(x) := \varphi_-(-\alpha, \beta, \gamma, -x) \quad (\text{respectively, } \widetilde{\varphi}_+(x) := -\varphi_-(-\alpha, \beta, -\gamma, -x)),$$

which is holomorphic in $D^{1/4}_{-\infty}(-r)$.

Theorem 2.4. *Suppose that* $\gamma = 0$. *Then (1.1) admits four solutions* $\varphi_0^{\pm}(x) := \pm\varphi_0(\alpha, \beta, x)$, $\widetilde{\varphi}_0^{\pm}(x) := \pm\varphi_0(-\alpha, \beta, -x)$ *with the properties:*

(a) $\varphi_0^{\pm}(x)$ *(respectively,* $\widetilde{\varphi}_0^{\pm}(x)$*) are holomorphic in the domain* $\operatorname{Re} x > r$ *(respectively,* $\operatorname{Re} x < -r$*), where* $r > 0$ *is sufficiently large;*
(b) $\varphi_0(\alpha, \beta, x)$ *(respectively,* $\varphi_0(-\alpha, \beta, -x)$*) admits the asymptotic representation*

$$\varphi_0(\alpha, \beta, x) \sim -\Phi_-(\alpha, \beta, 0, x) = \Phi_+(\alpha, \beta, 0, x)$$
$$(\text{respectively, } \varphi_0(-\alpha, \beta, -x) \sim -\Phi_-(-\alpha, \beta, 0, -x) = \Phi_+(-\alpha, \beta, 0, -x))$$

as $x \to \infty$ *through the domain* $\operatorname{Re} x > r$ *(respectively,* $\operatorname{Re} x < -r$*).*

Since $\bigcup_{n=0}^{\infty} \{x - n \,|\, x \in D^{1/4}_{+\infty}(r)\} = \bigcup_{n=0}^{\infty} \{x + n \,|\, x \in D^{1/4}_{-\infty}(-r)\} = \mathbb{C}$, by virtue of equation (1.1), these asymptotic solutions are continued meromorphically to the whole complex plane. Thus we have

Theorem 2.5. *If* $\gamma^2/\alpha \in \mathbb{R}$, *then (1.1) admits a nontrivial meromorphic solution in the whole complex plane.*

2.2. Equation (1.2)

If $\gamma = 0$, then (1.2) coincides with (1.1). We may suppose that $\gamma \neq 0$.

Proposition 2.6. *Equation (1.2) admits formal solutions of the form*

$$\Phi^*_{\pm}(\alpha, \beta, \gamma, x) = \pm\sqrt{\alpha/2}x^{1/2} + \sum_{j\geqslant 1} c_j^{*\pm} x^{-j/2},$$

where $c_j^{*\pm}$ $(j = 1, 2, 3, \ldots)$ *are rational functions of* $\sqrt{\alpha/2}$, β *and* γ, *in particular*

$$c_1^{*\pm} = \pm\frac{\beta}{4\sqrt{\alpha/2}}, \qquad c_2^{*\pm} = \frac{\gamma}{2\alpha}.$$

These formal solutions satisfy $\Phi^*_-(\alpha, \beta, -\gamma, x) = -\Phi^*_+(\alpha, \beta, \gamma, x)$.

Theorem 2.7. *Suppose that* $\lambda^* := 2\gamma/(\alpha\sqrt{\alpha/2}) \in \mathbb{R}\backslash\{0\}$.
(1) If $\lambda^* > 0$, *then (1.2) has a solution* $\varphi^*_-(x) = \varphi^*_-(\alpha, \beta, \gamma, x)$ *with the properties:*
(a) $\varphi^*_-(x)$ *is holomorphic in the domain* $D^{3/4}_{+\infty}(r)$*;*
(b) $\varphi^*_-(x)$ *admits the asymptotic representation*

$$\varphi^*_-(x) \sim \Phi^*_-(\alpha, \beta, \gamma, x)$$

as $x \to \infty$ *through* $D^{3/4}_{+\infty}(r)$.
(2) If $\lambda^* < 0$, *then (1.2) has the solution* $\varphi^*_+(x) := -\varphi^*_-(\alpha, \beta, -\gamma, x)$ *holomorphic in* $D^{3/4}_{+\infty}(r)$.

Theorem 2.8. *Suppose that* $\tilde{\lambda}^* := -2\gamma/(\alpha\sqrt{-\alpha/2}) \in \mathbb{R}\backslash\{0\}$. *If* $\tilde{\lambda}^* > 0$ *(respectively,* $\tilde{\lambda}^* < 0$*), then (1.2) has the solution*

$$\widetilde{\varphi}^*_-(x) := \varphi^*_-(-\alpha, \beta, \gamma, -x) \quad (\textit{respectively}, \ \widetilde{\varphi}^*_+(x) := -\varphi^*_-(-\alpha, \beta, -\gamma, -x)),$$

which is holomorphic in $D^{3/4}_{-\infty}(-r)$.

Theorem 2.9. *If* $\gamma^2/\alpha^3 \in \mathbb{R}$, *then (1.2) admits a nontrivial meromorphic solution in the whole complex plane.*

2.3. Equation (1.3)

Our results on (1.3) are stated as follows.

Proposition 2.10. *Equation (1.3) admits formal solutions of the form*

$$\Psi_\pm(\alpha, \beta, \gamma, x) = \pm\sqrt{-\alpha/2}x^{1/2} + \sum_{j\geqslant 1} d^\pm_j x^{-j/2},$$

where $d^\pm_j$ $(j = 1, 2, 3, \ldots)$ *are rational functions of* $\sqrt{-\alpha/2}$, β *and* γ, *in particular*

$$d^\pm_1 = \pm\frac{2-\beta}{4\sqrt{-\alpha/2}}, \qquad d^\pm_2 = \frac{\gamma}{2\alpha}.$$

These formal solutions satisfy $\Psi_-(\alpha, \beta, -\gamma, x) = -\Psi_+(\alpha, \beta, \gamma, x)$.

Theorem 2.11. *If* $\alpha > 0$, *then (1.3) has two solutions* $\psi_-(x) = \psi_-(\alpha, \beta, \gamma, x)$ *and* $\psi_+(x) := -\psi_-(\alpha, \beta, -\gamma, x)$ *with the properties:*
(a) $\psi_\pm(x)$ *are holomorphic in the domain* $D^{1/2}_{+\infty}(r)$*;*
(b) $\psi_\pm(x)$ *admit the asymptotic representations*

$$\psi_\pm(x) \sim \Psi_\pm(\alpha, \beta, \gamma, x)$$

as $x \to \infty$ *through* $D^{1/2}_{+\infty}(r)$.

Theorem 2.12. *If* $\alpha < 0$, *then (1.3) has two solutions* $\widetilde{\psi}_\pm(x) := \mp\psi_-(-\alpha, \beta, \mp\gamma, -x)$ *holomorphic in* $D^{1/2}_{-\infty}(-r)$.

Theorem 2.13. *If* $\alpha \in \mathbb{R}\backslash\{0\}$, *then (1.3) admits a nontrivial meromorphic solution in the whole complex plane.*

3. Proof of proposition 2.1

To prove proposition 2.1 we construct formal series (2.1). In this section we treat formal power series in $x^{-1/2}$. We write $f(x) = O_{(-d)}[x^{d_0}]$ $(d_0, d \in \mathbb{Q}, d > 0)$ if $x^{-d_0}f(x) \in \mathbb{C}[[x^{-d}]]$, $\mathbb{C}[[X]]$ denoting the ring of formal power series in X.

Substitute $y(x) = cx^{1/2} + y_0(x)$ $(c \neq 0)$ into (1.1). Observing that $(x+1)^{1/2} + (x-1)^{1/2} = x^{1/2}(2 - x^{-2}/4 + O_{(-2)}[x^{-4}])$, we have

$$y_0(x+1) + y_0(x-1) = -cx^{1/2}\left(2 - \frac{x^{-2}}{4} + O_{(-2)}[x^{-4}]\right) + \gamma + \frac{\alpha}{c}x^{1/2}\left(1 + \frac{\beta}{\alpha}x^{-1}\right) + \sum_{j\geqslant 1}\frac{(-1)^j\alpha}{c^{j+1}}x^{-(j-1)/2}\left(1 + \frac{\beta}{\alpha}x^{-1}\right)y_0(x)^j,$$

whose right-hand member is convergent if $|x| > 1$ and $|y_0(x)| < |cx^{1/2}|$. Choose $c = c_\pm = \pm\sqrt{\alpha/2}$ so that the right-hand member becomes

$$\gamma + \frac{\beta}{c_\pm}x^{-1/2} + \frac{c_\pm}{4}x^{-3/2}(1 + O_{(-2)}[x^{-2}]) + \sum_{j\geqslant 1}\frac{(-1)^j\alpha}{c_\pm^{j+1}}x^{-(j-1)/2}\left(1 + \frac{\beta}{\alpha}x^{-1}\right)y_0(x)^j.$$

To remove the constant term γ we put $y_0(x) = c_0 + y_1(x)$. Comparing the constant terms on both sides, we have $2c_0 = \gamma - \alpha c_0\big/c_\pm^2 = \gamma - 2c_0$, that is, $c_0 = \gamma/4$. Then the equation is written in the form

$$y_1(x+1) + y_1(x-1) = f_0^\pm(x) + \big(-2 + f_1^\pm(x)\big)y_1(x) + \sum_{j\geqslant 2} f_j^\pm(x)y_1(x)^j, \tag{3.1}$$

where

$$f_0^\pm(x) = \frac{8\beta+\gamma^2}{8c_\pm}x^{-1/2} + O_{(-1/2)}[x^{-1}],$$
$$f_1^\pm(x) = \frac{\gamma}{c_\pm}x^{-1/2} - \frac{8\beta+3\gamma^2}{4\alpha}x^{-1} + O_{(-1/2)}[x^{-3/2}],$$
$$f_2^\pm(x) = \frac{2}{c_\pm}x^{-1/2} + O_{(-1/2)}[x^{-1}], \qquad f_j^\pm(x) = O_{(-1/2)}[x^{-(j-1)/2}] \qquad (j \geqslant 3).$$

Note that $f_j^\pm(x)$ $(j \geqslant 0)$ are holomorphic for $|x| > 1$, and that the right-hand member is convergent if $|x| > 1$ and if $|x^{-1/2}y_1(x)|$ is sufficiently small.

In (3.1) put $y_1(x) = c_1x^{-1/2} + y_2(x)$ and compare the coefficients of $x^{-1/2}$. Choosing $c_1 = c_1^\pm = (8\beta+\gamma^2)c_\pm^{-1}\big/32$, we have

$$y_2(x+1) + y_2(x-1) = f_0^{*\pm}(x) + \big(-2 + f_1^{*\pm}(x)\big)y_2(x) + \sum_{j\geqslant 2} f_j^{*\pm}(x)y_2(x)^j,$$

with

$$f_0^{*\pm}(x) = O_{(-1/2)}[x^{-1}],$$
$$f_1^{*\pm}(x) = \frac{\gamma}{c_\pm}x^{-1/2} - \frac{\gamma^2}{2\alpha}x^{-1} + O_{(-1/2)}[x^{-3/2}],$$
$$f_j^{*\pm}(x) = O_{(-1/2)}[x^{-(j-1)/2}] \qquad (j \geqslant 2).$$

Repeating suitably chosen substitutions of the form $y_j(x) = c_j^\pm x^{-j/2} + y_{j+1}(x)$ with $c_j^\pm$ rational in $(\sqrt{\alpha/2}, \beta, \gamma)$ infinitely many times, we obtain the formal transformations

$$y(x) = \Phi_\pm(\alpha,\beta,\gamma,x) + z(x), \qquad \Phi_\pm(\alpha,\beta,\gamma,x) = \pm\sqrt{\alpha/2}x^{1/2} + \frac{\gamma}{4} + \sum_{j\geqslant 1} c_j^\pm x^{-j/2},$$

by which (1.1) is taken into

$$z(x+1) + (2 + \hat{g}_\pm(x))z(x) + z(x-1) = \sum_{j\geqslant 2}\hat{g}_j^\pm(x)z(x)^j \tag{3.2}$$

with

$$\hat{g}_\pm(x) = \mp\lambda x^{-1/2} + \frac{\lambda^2}{4}x^{-1} + O_{(-1/2)}[x^{-3/2}], \qquad \lambda = \frac{\gamma}{\sqrt{\alpha/2}},$$
$$\hat{g}_j^\pm(x) = O_{(-1/2)}[x^{-(j-1)/2}] \qquad (j \geqslant 2). \tag{3.3}$$

Since (3.2) admits the trivial solution $z(x) \equiv 0$, the formal series $\Phi_\pm(\alpha,\beta,\gamma,x)$ satisfy (1.1). By the procedure above the coefficients of each formal solution are uniquely determined. Observing that equation (1.1) remains invariant under the substitution $\gamma \mapsto -\gamma$,

$y(x) \mapsto -y(x)$, we obtain the relation between $\Phi_-(\alpha, \beta, -\gamma, x)$ and $\Phi_+(\alpha, \beta, \gamma, x)$ as in proposition 2.1.

Remark 3.1. In particular, if $\gamma = 0$, that is, $\lambda = 0$, then the transformations above are

$$y_0(x) = y_1(x), \qquad y_1(x) = \frac{\beta}{4c_\pm}x^{-1/2} + y_2(x), \qquad y_2(x) = \frac{(\alpha^2 - 2\beta^2)}{32\alpha c_\pm}x^{-3/2} + y_3(x), \ldots,$$

which yield (3.2) with

$$\hat{g}_\pm(x) = -\frac{x^{-2}}{4} + O_{(-1/2)}[x^{-3}]. \tag{3.4}$$

4. Difference equation equivalent to (1.1)

Recall the formal series

$$\Phi_-(\alpha, \beta, \gamma, x) = -\sqrt{\alpha/2}x^{1/2} + \frac{\gamma}{4} + \sum_{j\geqslant 1} c_j^- x^{-j/2}$$

obtained in section 3. By [20, theorem 9.3], there exists a holomorphic function $\xi(x)$ in the half-plane Σ_0 : $\mathrm{Re}\, x > r_0$ such that

$$\xi(x) \sim \sum_{j\geqslant 1} c_j^- x^{-j/2}$$

as $x \to \infty$ through Σ_0, where r_0 is a sufficiently large positive number. Note that we may uniquely determine $C_j^\pm \in \mathbb{C}$ in such a way that the identities

$$\sum_{j\geqslant 1} c_j^-(x \pm 1)^{-j/2} = \sum_{j\geqslant 1} C_j^\pm x^{-j/2}$$

hold as formal series. The following lemma is easily checked.

Lemma 4.1. *The functions $\xi(x \pm 1)$ admit the asymptotic representations*

$$\xi(x \pm 1) \sim \sum_{j\geqslant 1} C_j^\pm x^{-j/2}$$

as $x \to \infty$ through the half-plane $\mathrm{Re}\, x > r_0 + 1$.

Observe that $|\xi(x)| = o(1)$ in Σ_0. Putting $y_1(x) = \xi(x) + z(x)$ in equation (3.1) with $f_j^-(x)$ and using lemma 4.1, we obtain the following difference equation:

Proposition 4.2. *By the transformation*

$$y(x) = -\sqrt{\alpha/2}x^{1/2} + \frac{\gamma}{4} + \xi(x) + z(x)$$

equation (1.1) is changed into

$$z(x + 1) + (2 + g(x))z(x) + z(x - 1) = G(x, z(x)) \tag{4.1}$$

with the properties:

(1) $g(x)$ is holomorphic in the half-plane Σ_1 : $\mathrm{Re}\, x > r_1$, and is expressed in the form

$$g(x) = \lambda x^{-1/2} + \frac{\lambda^2}{4}x^{-1} + O(x^{-3/2}), \qquad \lambda = \frac{\gamma}{\sqrt{\alpha/2}} \tag{4.2}$$

as $x \to \infty$ *through* Σ_1*, in particular, if* $\lambda = 0$,

$$g(x) = -\frac{x^{-2}}{4} + O(x^{-3}), \tag{4.3}$$

where $r_1 > r_0 > 0$ *is sufficiently large;*

(2) $G(x, \zeta)$ *is holomorphic in the domain* $\Sigma_1 \times \{\zeta \,|\, |\zeta| < 1\}$ *and is expanded into the convergent series*

$$G(x, \zeta) = g_0(x) + \sum_{j \geqslant 2} g_j(x)\zeta^j,$$

whose coefficients satisfy

$$g_0(x) \sim 0, \qquad g_j(x) = O(x^{-(j-1)/2}) \qquad (j \geqslant 2) \tag{4.4}$$

as $x \to \infty$ *through* Σ_1.

Remark 4.1. The coefficients above admit the asymptotic representations

$$g(x) \sim \hat{g}_-(x), \qquad g_j(x) \sim \hat{g}_j^-(x) \qquad (j \geqslant 2)$$

as $x \to \infty$ through Σ_1, where $\hat{g}_-(x)$ and $\hat{g}_j^-(x)$ are the formal series given by (3.3).

5. Associated linear system

In $\Sigma_1 : \mathrm{Re}\, x > r_1$, consider the equation

$$z(x+1) + (2 + g(x))z(x) + z(x-1) = 0 \tag{5.1}$$

corresponding to the linear part of (4.1). Then $u(x) = \mathrm{e}^{-\pi \mathrm{i} x} z(x)$ satisfies

$$u(x+1) - (2 + g(x))u(x) + u(x-1) = 0, \tag{5.2}$$

which is equivalent to the system of difference equations

$$\mathbf{u}(x+1) = A(x)\mathbf{u}(x),$$
$$A(x) = \begin{pmatrix} 1 + g(x) & 1 \\ g(x) & 1 \end{pmatrix}, \qquad \mathbf{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \tag{5.3}$$

with $u_1(x) = u(x)$, $u_2(x) = u(x) - u(x-1)$.

In this section we construct a fundamental matrix solution for system (5.3). The following matrix basis is very effective in carrying out our computations:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad K := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad L := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which satisfies

$$J^2 = -K^2 = L^2 = I,$$
$$JK = -KJ = -L, \qquad KL = -LK = -J, \qquad LJ = -JL = K.$$

The process of construction is divided into several steps. First we treat the case where $\gamma \neq 0$, that is, $\lambda \neq 0$.

5.1. Step 1

We begin with the following lemma, which is checked by a direct computation.

Lemma 5.1. *The matrix $A(x)$ admits the eigenvalues*

$$\rho_\pm(x) := 1 + g(x)/2 \pm \sqrt{g(x) + g(x)^2/4},$$

where the branch of the square root is fixed as $x \to \infty$ through Σ_1. Moreover

$$T(x)^{-1}A(x)T(x) = \begin{pmatrix} \rho_-(x) & 0 \\ 0 & \rho_+(x) \end{pmatrix} = \left(1 + \frac{g(x)}{2}\right)I + \sqrt{g(x) + \frac{g(x)^2}{4}}\,J,$$

where

$$T(x) = \begin{pmatrix} 1 & 1 \\ -g(x)/2 - \sqrt{g(x)+g(x)^2/4} & -g(x)/2 + \sqrt{g(x)+g(x)^2/4} \end{pmatrix}.$$

By this lemma we have

$$T(x+1)^{-1}T(x) = I + h_1(x)(I - L) + h_2(x)(K - J), \tag{5.4}$$

where

$$\begin{aligned} h_1(x) &= \frac{2h_2(x)(1 + (g(x) + g(x+1))/4)}{\sqrt{g(x)+g(x)^2/4} + \sqrt{g(x+1)+g(x+1)^2/4}}, \\ h_2(x) &= \frac{g(x) - g(x+1)}{4\sqrt{g(x+1)+g(x+1)^2/4}}. \end{aligned} \tag{5.5}$$

Observe that, by (4.2),

$$\sqrt{g(x)+g(x)^2/4} = \lambda^{1/2}x^{-1/4}(1 + \lambda x^{-1/2}/4 + O(x^{-1})) \tag{5.6}$$

as $x \to \infty$ through $\Sigma_1 : \mathrm{Re}\, x > r_1$, with the branch of $x^{1/4}$ taken so that $x^{1/4} > 0$ for $x > 0$. Here, and in some sequential steps, the constant r_1 is again chosen larger if necessary. Then these quantities are estimated as follows:

$$h_1(x) = \frac{x^{-1}}{8}(1 + O(x^{-1/2})), \qquad h_2(x) = O(x^{-5/4}) \tag{5.7}$$

as $x \to \infty$ through Σ_1. By $\mathbf{u}(x) = T(x)\mathbf{v}(x)$, system (5.3) is taken into

$$\mathbf{v}(x+1) = B(x)\mathbf{v}(x) \tag{5.8}$$

with

$$B(x) = T(x+1)^{-1}A(x)T(x) = T(x+1)^{-1}T(x)\begin{pmatrix} \rho_-(x) & 0 \\ 0 & \rho_+(x) \end{pmatrix}.$$

Using (5.4) and lemma 5.1, we have

$$\begin{aligned} B(x) &= b_0(x)I + b_1(x)J + b_2(x)K + b_3(x)L, \\ b_0(x) &= (1 + h_1(x))(1 + g(x)/2) - h_2(x)\sqrt{g(x)+g(x)^2/4}, \\ b_1(x) &= (1 + h_1(x))\sqrt{g(x)+g(x)^2/4} - h_2(x)(1 + g(x)/2), \\ b_2(x) &= h_2(x)(1 + g(x)/2) - h_1(x)\sqrt{g(x)+g(x)^2/4}, \\ b_3(x) &= -h_1(x)(1 + g(x)/2) + h_2(x)\sqrt{g(x)+g(x)^2/4}. \end{aligned}$$

By (5.6) and (5.7),

$$\begin{aligned} b_0(x) &= 1 + \frac{\lambda}{2}x^{-1/2} + \frac{1+\lambda^2}{8}x^{-1} + O(x^{-3/2}), \\ b_1(x) &= \lambda^{1/2}x^{-1/4} + \frac{\lambda^{3/2}}{4}x^{-3/4} + O(x^{-5/4}), \\ b_2(x) &= O(x^{-5/4}), \\ b_3(x) &= -\frac{x^{-1}}{8} + O(x^{-3/2}) \end{aligned} \tag{5.9}$$

as $x \to \infty$ through Σ_1.

5.2. Step 2

To remove the term $-x^{-1}/8$ of $b_3(x)$, we apply a further transformation $\mathbf{v}(x) = (I + p(x)K)\mathbf{y}(x)$, which changes (5.8) into

$$\mathbf{y}(x+1) = E(x)\mathbf{y}(x) \tag{5.10}$$

with

$$\begin{aligned} E(x) &= (I + p(x+1)K)^{-1}B(x)(I + p(x)K) \\ &= \frac{1}{1+p(x+1)^2}(I - p(x+1)K)B(x)(I + p(x)K) \\ &= \frac{1}{1+p(x+1)^2}[((1 + p(x)p(x+1))b_0(x) + (p(x+1) - p(x))b_2(x))I \\ &\quad + ((1 - p(x)p(x+1))b_1(x) + (p(x+1) + p(x))b_3(x))J \\ &\quad + ((1 + p(x)p(x+1))b_2(x) - (p(x+1) - p(x))b_0(x))K \\ &\quad + ((1 - p(x)p(x+1))b_3(x) - (p(x+1) + p(x))b_1(x))L]. \end{aligned} \tag{5.11}$$

Put $p(x) = -\lambda^{-1/2}x^{-3/4}/16$. Then

$$\begin{aligned} b_3(x) - (p(x+1) + p(x))b_1(x) &= -\frac{x^{-1}}{8}(1 + O(x^{-1/2})) \\ &\quad - \lambda^{1/2}x^{-1/4}(1 + O(x^{-1/2}))(p(x+1) + p(x)) = O(x^{-3/2}), \end{aligned}$$

and hence

$$\begin{aligned} E(x) &= (e_0(x) + x^{-1}/8)I + e_1(x)J + O(x^{-5/4})K + O(x^{-3/2})L, \\ e_0(x) &= 1 + \frac{\lambda}{2}x^{-1/2} + \frac{\lambda^2}{8}x^{-1} + O(x^{-3/2}), \\ e_1(x) &= \lambda^{1/2}x^{-1/4} + \frac{\lambda^{3/2}}{4}x^{-3/4} + O(x^{-5/4}). \end{aligned} \tag{5.12}$$

5.3. Step 3

Set $\sigma(x) = (4/3)\lambda^{1/2}x^{3/4} + (1/3)\lambda^{3/2}x^{1/4}$. Then

$$\mathrm{e}^{-\sigma(x+1)+\sigma(x)} = e_0(x) - e_1(x) + O(x^{-5/4}), \qquad \mathrm{e}^{\sigma(x+1)-\sigma(x)} = e_0(x) + e_1(x) + O(x^{-5/4}).$$

The function $\sigma(x)$ is easily obtained by supposing that $\sigma(x) = \sigma_1 x^{3/4} + \sigma_2 x^{1/4}$ and comparing the coefficients on both sides. Moreover, $(x+1)^{1/8} = x^{1/8}(1 + x^{-1}/8 + O(x^{-2}))$. Hence by

$$\mathbf{y}(x) = x^{1/8}\begin{pmatrix} \mathrm{e}^{-\sigma(x)} & 0 \\ 0 & \mathrm{e}^{\sigma(x)} \end{pmatrix}\mathbf{w}(x)$$

system (5.10) is reduced to

$$\mathbf{w}(x+1) = F(x)\mathbf{w}(x) \tag{5.13}$$

with

$$F(x) = I + \begin{pmatrix} \varepsilon_{11}(x) & \mathrm{e}^{2\sigma(x)}\varepsilon_{12}(x) \\ \mathrm{e}^{-2\sigma(x)}\varepsilon_{21}(x) & \varepsilon_{22}(x) \end{pmatrix}, \qquad \varepsilon_{ij}(x) = O(x^{-5/4}).$$

5.4. Step 4

To estimate a fundamental matrix solution of (5.13) we use the following lemma.

Lemma 5.2. *Let $S(x)$ be a matrix given by $S(x) = I + X(x)$, $X(x) = O(x^{-1-\tau})$ with some $\tau > 0$ as $x \to \infty$ through $D^a_{+\infty}(r)$ $(a > 0, r > 0)$. Here $X(x) = O(x^{-1-\tau})$ means that every entry of $X(x)$ is $O(x^{-1-\tau})$. Then the infinite product*

$$\Pi_\infty(x) = S(x)S(x+1)\cdots S(x+n)\cdots \qquad (n \in \mathbb{N})$$

is holomorphic in $D^a_{+\infty}(r)$ and satisfies $\Pi_\infty(x) = I + O(x^{-\tau})$.

Proof. Let m and n be arbitrary positive integers satisfying $n > m$, and write $\Pi_n(x) := S(x)S(x+1)\cdots S(x+n)$. Then

$$\begin{aligned}
\|\Pi_n(x) - \Pi_m(x)\| &= \|\Pi_m(x)(S(x+m+1)\cdots S(x+n) - I)\| \\
&\leqslant \|\Pi_m(x)\| \cdot \|(I + X(x+m+1))\cdots(I + X(x+n)) - I\| \\
&\leqslant \prod_{j=0}^{m}(1 + C_0|x+j|^{-1-\tau}) \cdot \left(\prod_{j=m+1}^{n}(1 + C_0|x+j|^{-1-\tau}) - 1\right) \\
&\leqslant \exp\left(C_0\sum_{j=0}^{m}|x+j|^{-1-\tau}\right)\left(\exp\left(C_0\sum_{j=m+1}^{n}|x+j|^{-1-\tau}\right) - 1\right)
\end{aligned}$$

in $D^a_{+\infty}(r)$. Here C_0 is some positive number, and $\|\cdot\|$ denotes the standard norm for matrices (see, for example [20, p 11]). Since $\sum_{j=0}^{\infty}|x+j|^{-1-\tau} = O(x^{-\tau})$ in $D^a_{+\infty}(r)$ (see, for example, [2, p 24]), the matrix function $\Pi_n(x)$ converges to $\Pi_\infty(x)$ uniformly for $x \in D^a_{+\infty}(r)$, which is holomorphic in $D^a_{+\infty}(r)$. Observing that, for every $n \in \mathbb{N}$,

$$\|\Pi_n(x) - I\| \leqslant \prod_{j=0}^{\infty}(1 + C_0|x+j|^{-1-\tau}) - 1 = O(x^{-\tau}),$$

we obtain $\Pi_\infty(x) = I + O(x^{-\tau})$, which completes the proof. □

Remark 5.1. In the statement of lemma 5.2, the domain $D^a_{+\infty}(r)$ may be replaced with the half-plane $\mathrm{Re}\, x > r$.

Now we are ready to estimate a fundamental matrix solution of (5.13). Suppose that $\lambda < 0$. In $D^{1/4}_{+\infty}(r_1)$, observe that

$$\begin{aligned}
x^{3/4} &= (\mathrm{Re}\, x + \mathrm{i}\,\mathrm{Im}\, x)^{3/4} = (\mathrm{Re}\, x)^{3/4}(1 + O(|\mathrm{Im}\, x|/|\mathrm{Re}\, x|)) \\
&= (\mathrm{Re}\, x)^{3/4}(1 + O(|\mathrm{Re}\, x|^{-3/4})) = (\mathrm{Re}\, x)^{3/4} + O(1),
\end{aligned}$$

and that $x^{1/4} = (\mathrm{Re}\, x)^{1/4} + O(1)$. Since $\lambda^{1/2}$ is pure imaginary,

$$\mathrm{e}^{\pm\sigma(x)} = O(1) \qquad \text{uniformly for } x \in D^{1/4}_{+\infty}(r_1), \tag{5.14}$$

so that $F(x) = I + O(x^{-5/4})$ in $D^{1/4}_{+\infty}(r_1)$. By lemma 5.2, in $D^{1/4}_{+\infty}(r_2)$,

$$W(x) := F(x)^{-1}F(x+1)^{-1}\cdots F(x+n)^{-1}\cdots = I + O(x^{-1/4})$$

is a fundamental matrix solution of (5.13), where r_2 $(> r_1)$ is a sufficiently large positive number. Observing that

$$\begin{pmatrix} \mathrm{e}^{-\sigma(x)} & 0 \\ 0 & \mathrm{e}^{\sigma(x)} \end{pmatrix} W(x) = (I + O(x^{-1/4}))\begin{pmatrix} \mathrm{e}^{-\sigma(x)} & 0 \\ 0 & \mathrm{e}^{\sigma(x)} \end{pmatrix},$$

we obtain the following:

Proposition 5.3. *If* $\lambda < 0$, *then system (5.3) admits the fundamental matrix solution*

$$U(x) = \begin{pmatrix} 1 + O(x^{-1/4}) & 1 + O(x^{-1/4}) \\ -\lambda^{1/2}x^{-1/4}(1 + O(x^{-1/4})) & \lambda^{1/2}x^{-1/4}(1 + O(x^{-1/4})) \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\sigma(x)} & 0 \\ 0 & \mathrm{e}^{\sigma(x)} \end{pmatrix} x^{1/8},$$

$$\sigma(x) = \frac{4}{3}\lambda^{1/2}x^{3/4} + \frac{\lambda^{3/2}}{3}x^{1/4}$$

as $x \to \infty$ *through* $D^{1/4}_{+\infty}(r_2)$. *Moreover,* $\det U(x) = 2\lambda^{1/2}(1 + O(x^{-1/4}))$.

5.5. *Case where* $\lambda = 0$

If $\lambda = 0$, then $g(x) = -x^{-2}/4 + O(x^{-3})$ in Σ_1 (cf (4.3)). From (5.5) we have $h_1(x) = (x^{-1}/2)(1 + O(x^{-1}))$ and $h_2(x) = O(x^{-2})$. Hence, by $\mathbf{u}(x) = T(x)\mathbf{v}(x)$, system (5.3) is changed into (5.8) with

$$B(x) = \left(1 + \frac{x^{-1}}{2} + O(x^{-2})\right)I + \left(\frac{\mathrm{i}}{2}x^{-1} + O(x^{-2})\right)J + O(x^{-2})K - \left(\frac{x^{-1}}{2} + O(x^{-2})\right)L,$$

and, by $\mathbf{v}(x) = (-J + \mathrm{i}K)\mathbf{y}(x)$, this is changed into

$$\mathbf{y}(x+1) = E_0(x)\mathbf{y}(x), \qquad E_0(x) = I + \begin{pmatrix} 1/2 & 1 \\ 0 & 1/2 \end{pmatrix} x^{-1} + O(x^{-2}).$$

The further transformation

$$\mathbf{y}(x) = \begin{pmatrix} 1 & \log x \\ 0 & 1 \end{pmatrix} x^{1/2}\mathbf{w}(x)$$

reduces this into the system $\mathbf{w}(x+1) = (I + O(x^{-2}))\mathbf{w}(x)$. Using lemma 5.2 together with remark 5.1, we have the following:

Proposition 5.4. *If* $\lambda = 0$, *then system (5.3) admits the fundamental matrix solution*

$$U(x) = \begin{pmatrix} 1 - \mathrm{i} + O(x^{-1}) & -1 + \mathrm{i} + O(x^{-1}) \\ -\frac{\mathrm{i}}{2}(1+\mathrm{i})x^{-1}(1 + O(x^{-1})) & -\frac{\mathrm{i}}{2}(1+\mathrm{i})x^{-1}(1 + O(x^{-1})) \end{pmatrix}$$
$$\times (I + O(x^{-1}(\log x)^2))\begin{pmatrix} 1 & \log x \\ 0 & 1 \end{pmatrix} x^{1/2}$$

as $x \to \infty$ *through* $\Sigma_2 : \mathrm{Re}\, x > r_2$. *Moreover,* $\det U(x) = -2\mathrm{i}(1 + O(x^{-1}(\log x)^2))$.

6. Proofs of the results on (1.1)

6.1. Proof of theorem 2.2

Suppose that $\lambda < 0$. To show the asymptotic validity of $\varphi_-(x)$, it is sufficient to prove that (4.1) has a solution such that $z(x) \sim 0$ as $x \to \infty$ through $D^{1/4}_{+\infty}(r)$ for some $r > 0$. Recall linear equation (5.1) corresponding to (4.1). By proposition 5.3, (5.1) admits the linearly independent solutions

$$z_1(x) = e^{\pi ix - \sigma(x)} x^{1/8}(1 + O(x^{-1/4})), \qquad z_2(x) = e^{\pi ix + \sigma(x)} x^{1/8}(1 + O(x^{-1/4}))$$

as $x \to \infty$ through $D^{1/4}_{+\infty}(r_2)$. The Casorati determinant is

$$\Delta(x) := \begin{vmatrix} z_1(x) & z_2(x) \\ z_1(x+1) & z_2(x+1) \end{vmatrix} = -e^{2\pi ix} \det U(x+1) = -2\lambda^{1/2} e^{2\pi ix}(1 + O(x^{-1/4})).$$

Note that, in $D^{1/4}_{+\infty}(r_2)$, every solution of the equation

$$\omega(x) = \mathcal{S}(x, \omega(x)) := \sum_{j=0}^{\infty} H(j; x) G(x + j, \omega(x + j)) \tag{6.1}$$

with

$$H(j; x) := \frac{z_1(x) z_2(x+j) - z_1(x+j) z_2(x)}{\Delta(x+j)}$$

satisfies (4.1), provided that the right-hand member converges. This summation relation is the difference version of an integral equation. We would like to construct a solution of (6.1) such that $\omega(x) \sim 0$ as $x \to \infty$ through $D^{1/4}_{+\infty}(r)$. By proposition 4.2, we may suppose that $G(x, \zeta)$ has the following properties:

(a) for $\operatorname{Re} x > r_2$, $|\zeta| < 1/2$,

$$|G(x, \zeta)| \leqslant |g_0(x)| + |\zeta|^2; \tag{6.2}$$

(b) for $\operatorname{Re} x > r_2$, $|\zeta_1|, |\zeta_2| < 1/2$,

$$|G(x, \zeta_2) - G(x, \zeta_1)| \leqslant |\zeta_2 - \zeta_1|(|\zeta_1| + |\zeta_2|). \tag{6.3}$$

By virtue of (5.14) there exists a positive number κ_0 such that, for every $j \in \mathbb{N} \cup \{0\}$,

$$|H(j; x)| \leqslant \kappa_0 |x + j|^{1/4} \tag{6.4}$$

uniformly for $x \in D^{1/4}_{+\infty}(r_2)$.

Let N be a given integer such that $N \geqslant 4$. Since $g_0(x) \sim 0$ as $x \to \infty$ through the domain $\operatorname{Re} x > r_2$, there exists a positive number M_N such that

$$|g_0(x)| \leqslant M_N |x|^{-N-3} \tag{6.5}$$

for $\operatorname{Re} x > r_2$. The following proposition guarantees the existence of an iterative sequence.

Proposition 6.1. *Suppose that r_N ($N \geqslant 4$) satisfies $r_N > r_2 + 1, 4M_N r_N^{-N+3} \leqslant 1$ and $\kappa_0 \kappa_{N+1} r_N^{-1} \leqslant 1$, where κ_{N+1} is a positive number such that $\sum_{j=0}^{\infty} |x + j|^{-N-2} \leqslant \kappa_{N+1} |x|^{-N-1}$ uniformly for* $\operatorname{Re} x > r_2$. *Then, we may define a sequence $\{\omega_n(x) | n = 0, 1, 2, \ldots\}$ by*

$$\omega_0(x) \equiv 0, \qquad \omega_{n+1}(x) = \mathcal{S}(x, \omega_n(x))$$

for $x \in D^{1/4}_{+\infty}(r_N)$. Furthermore, for every n, $\omega_n(x)$ is holomorphic in $D^{1/4}_{+\infty}(r_N)$, and

$$|\omega_n(x)| \leqslant 2M_N |x|^{-N}. \tag{6.6}$$

Proof. This proposition is verified by induction on n. Suppose that $\omega_n(x)$ is holomorphic in $D^{1/4}_{+\infty}(r_N)$ and satisfies (6.6). Then, using (6.4), (6.2) and (6.5), we have, for every $x \in D^{1/4}_{+\infty}(r_N)$ and for every $j \geqslant 0$,

$$|H(j;x)G(x+j,\omega_n(x+j))| \leqslant |H(j;x)|(|g_0(x+j)| + |\omega_n(x+j)|^2)$$
$$\leqslant \kappa_0|x+j|^{1/4}(M_N|x+j|^{-N-3} + 4M_N^2|x+j|^{-2N})$$
$$\leqslant 2\kappa_0 M_N|x+j|^{-N-3+1/4} \leqslant 2\kappa_0 M_N|x+j|^{-N-2} \leqslant 2\kappa_0 M_N(r_2+j)^{-N-2},$$

which implies the uniform convergence of the summation $\mathcal{S}(x,\omega_n(x))$. Hence $\omega_{n+1}(x) = \mathcal{S}(x,\omega_n(x))$ is holomorphic in $D^{1/4}_{+\infty}(r_N)$, and

$$|\omega_{n+1}(x)| = |\mathcal{S}(x,\omega_n(x))| \leqslant 2\kappa_0 M_N \sum_{j=0}^{\infty} |x+j|^{-N-2}$$
$$\leqslant 2\kappa_0\kappa_{N+1} M_N|x|^{-N-1} \leqslant 2M_N|x|^{-N},$$

implying that $\omega_{n+1}(x)$ satisfies (6.6). This completes the proof. □

For every $n \geqslant 1$, we have

$$|\omega_{n+1}(x) - \omega_n(x)| \leqslant 4\kappa_0\kappa_{N-2}M_N r^{-1} \sup_{0\leqslant j<\infty} |\omega_n(x+j) - \omega_{n-1}(x+j)| \tag{6.7}$$

in $D^{1/4}_{+\infty}(r_N)$ (for κ_{N-2} see proposition 6.1, and note that $N \geqslant 4$). Indeed, by (6.3) and proposition 6.1, the left-hand member is estimated as follows:

$$|\mathcal{S}(x,\omega_n(x)) - \mathcal{S}(x,\omega_{n-1}(x))|$$
$$\leqslant \sum_{j=0}^{\infty} |H(j;x)||G(x+j,\omega_n(x+j)) - G(x+j,\omega_{n-1}(x+j))|$$
$$\leqslant \sum_{j=0}^{\infty} \kappa_0|x+j|^{1/4}\cdot 4M_N|x+j|^{-N}|\omega_n(x+j) - \omega_{n-1}(x+j)|$$
$$\leqslant 4\kappa_0\kappa_{N-2}M_N|x|^{-N+2} \sup_{0\leqslant j<\infty} |\omega_n(x+j) - \omega_{n-1}(x+j)|. \tag{6.8}$$

Rechoosing r_N so that $4\kappa_0\kappa_{N-2}M_N r_N^{-1} < 1/2$, if necessary, we have from (6.7) that

$$\sup_{0\leqslant j<\infty} |\omega_{n+1}(x+j) - \omega_n(x+j)| \leqslant 2^{-n} \sup_{0\leqslant j<\infty} |\omega_1(x+j)| \leqslant 2^{-n+1}M_N$$

in $D^{1/4}_{+\infty}(r_N)$. Hence $\omega_n(x)$ converges uniformly to a function $\omega(x)$ holomorphic in $D^{1/4}_{+\infty}(r_N)$, and by (6.6) we have

$$|\omega(x)| \leqslant 2M_N|x|^{-N} \tag{6.9}$$

in $D^{1/4}_{+\infty}(r_N)$. Furthermore, by the same argument as in the proof of proposition 6.1, we can verify the uniform convergence of the summation $\mathcal{S}(x,\omega(x))$ in $D^{1/4}_{+\infty}(r_N)$. Note that, in (6.8), $\omega_{n-1}(x)$ may be replaced with $\omega(x)$, because of (6.9). Thus we have

$$|\mathcal{S}(x,\omega(x)) - \mathcal{S}(x,\omega_n(x))| \leqslant \frac{1}{2} \sup_{0\leqslant j<\infty} |\omega(x+j) - \omega_n(x+j)|$$
$$\leqslant \frac{1}{2}\sum_{k\geqslant n} \sup_{0\leqslant j<\infty} |\omega_{k+1}(x+j) - \omega_k(x+j)| \leqslant M_N \sum_{k\geqslant n} 2^{-k} = 2^{-n+1}M_N,$$

so that $\mathcal{S}(x,\omega_n(x))$ converges to $\mathcal{S}(x,\omega(x))$ for $x \in D^{1/4}_{+\infty}(r_N)$. Therefore $\omega(x)$ satisfies equation (6.1). Now we set $r = r_4$. Since $\omega(x)$ is holomorphic in $D^{1/4}_{+\infty}(r)$, we deduce from (6.9) that, for each $N \geqslant 4$, $x^N\omega(x)$ is bounded in $D^{1/4}_{+\infty}(r)$. Consequently (4.1) has the solution $z(x) = \omega(x)$ such that $\omega(x) \sim 0$ as $x \to \infty$ through $D^{1/4}_{+\infty}(r)$. Thus we obtain assertion (1) of theorem 2.2.

Since (1.1) is invariant under the substitution $\gamma \mapsto -\gamma$, $y(x) \mapsto -y(x)$, assertion (2) immediately follows from assertion (1). This completes the proof of theorem 2.2.

6.2. Proofs of theorems 2.3 and 2.4

Equation (1.1) is invariant under the substitution $\alpha \mapsto -\alpha$, $x \mapsto -x$, $y(-x) \mapsto y(x)$. From theorem 2.2 together with this fact, we immediately obtain theorem 2.3.

In the case where $\lambda = 0$, by proposition 5.4, equation (5.1) admits the linearly independent solutions

$$z_1(x) = \mathrm{e}^{\pi \mathrm{i} x} x^{1/2}(1 + O(x^{-1}(\log x)^2)),$$
$$z_2(x) = \mathrm{e}^{\pi \mathrm{i} x} x^{1/2} \log x(1 + O(x^{-1}(\log x)^2)) - \mathrm{e}^{\pi \mathrm{i} x} x^{1/2}(1 + O(x^{-1}(\log x)^2)),$$

with the Casorati determinant $\Delta(x) := -\mathrm{e}^{2\pi \mathrm{i} x}(1 + O(x^{-1}(\log x)^2))$ in the domain $\mathrm{Re}\, x > r_2$. For $\mathrm{Re}\, x > r_2$, we have $|H(j; x)| \leqslant \kappa_0|x + j| \log |x + j|$ and $|g_0(x)| \leqslant M_N|x|^{-N-4}$ instead of (6.4) and (6.5), respectively, where N is a given integer such that $N \geqslant 5$. Choose r_N $(N \geqslant 5)$ satisfying $r_N > r_2 + 1$, $4M_N r_N^{-N+4} \leqslant 1$ and $\kappa_0\kappa_{N+1} r_N^{-1} \leqslant 1$. Then, in the domain $D^*_{+\infty}(r_N) := \{x | \mathrm{Re}\, x > r_2, |x| > r_N\}$, we may define the sequence $\{\omega_n(x) | n = 0, 1, 2, \ldots\}$ as in proposition 6.1, which satisfies $|\omega_n(x)| \leqslant 2M_N|x|^{-N}$ for $x \in D^*_{+\infty}(r_N)$. By the same argument as in section 6.1 with slight modification, we can show that (4.1) has a solution $z(x)$ satisfying $z(x) \sim 0$ as $x \to \infty$ through $D^*_{+\infty}(r_5)$. Thus we obtain the required solution

$$\varphi_0^-(x) = -\varphi_0(\alpha, \beta, x) \sim \Phi_-(\alpha, \beta, 0, x)$$

as $x \to \infty$ through the domain $\mathrm{Re}\, x > r$ $(= r_5)$. The other asymptotic solutions are derived by using the invariant properties of (1.1) under the substitutions $y(x) \mapsto -y(x)$ and $(\alpha, x, y(-x)) \mapsto (-\alpha, -x, y(x))$. In this way theorem 2.4 is obtained.

7. Proofs of the results on (1.2)

Putting $y(x) = c_\pm x^{1/2} + y_0(x)$, $c_\pm = \pm\sqrt{\alpha/2}$ in (1.2), we get an equation of the form

$$y_0(x+1) + y_0(x-1) = \frac{\beta}{c_\pm}x^{-1/2} + \frac{2\gamma}{\alpha}x^{-1} + \frac{c_\pm}{4}x^{-3/2} + O_{(-1/2)}[x^{-2}]$$
$$+ \left(1 + \frac{\beta}{\alpha}x^{-1}\right)\left(-2y_0(x) + \frac{2}{c_\pm}x^{-1/2}y_0(x)^2 - \frac{4}{\alpha}x^{-1}y_0(x)^3 + \cdots\right)$$
$$- \frac{4\gamma}{c_\pm\alpha}x^{-3/2}y_0(x) + \frac{12\gamma}{\alpha^2}x^{-2}y_0(x)^2 + \cdots,$$

whose right-hand member is convergent if $|x| > 1$ and $|y_0(x)| < |c_\pm x^{1/2}|$. Repeating the same argument as in the proof of proposition 2.1, we obtain the formal transformations

$$y(x) = c_\pm x^{1/2} + \frac{\beta}{4c_\pm}x^{-1/2} + \frac{\gamma}{2\alpha}x^{-1} + \frac{1}{4}\left(\frac{c_\pm}{4} - \frac{\beta^2}{4c_\pm\alpha}\right)x^{-3/2} + O_{(-1/2)}[x^{-2}] + z(x),$$

by which (1.2) is changed into

$$z(x+1) + z(x-1) = \left(-2 - \frac{2\gamma}{c_\pm\alpha}x^{-3/2} + \frac{x^{-2}}{4} + O_{(-1/2)}[x^{-5/2}]\right)z(x)$$
$$+ \frac{2}{c_\pm}x^{-1/2}(1 + O_{(-1/2)}[x^{-1}])z(x)^2 - \frac{4}{\alpha}x^{-1}(1 + O_{(-1/2)}[x^{-1}])z(x)^3 + \cdots$$

admitting the trivial solution $z(x) \equiv 0$. In this way we get the formal solutions of (1.2) as in proposition 2.6.

To prove theorem 2.7, putting $y(x) = -\sqrt{\alpha/2}x^{1/2} + \xi(x) + z(x)$ with $\xi(x) \sim \sum_{j\geqslant 1} c_j^{*-} x^{-j/2}$ in $\Sigma_1 : \mathrm{Re}\, x > r_1$, and using lemma 4.1, we obtain an equivalent difference equation of the form (4.1) with

$$g(x) = \lambda x^{-3/2} - \frac{x^{-2}}{4} + O(x^{-5/2}), \qquad \lambda := -\lambda^* = \frac{-2\gamma}{\alpha\sqrt{\alpha/2}} \tag{7.1}$$

as $x \to \infty$ through Σ_1, instead of (4.2), where r_1 is sufficiently large. The associated linear system is of the form (5.3) with $g(x)$ given by (7.1). Since $\sqrt{g(x) + g(x)^2/4} = \lambda^{1/2}x^{-3/4}(1 + O(x^{-1/2}))$, by (5.5) we have $h_1(x) = (3/8)x^{-1}(1 + O(x^{-1/2}))$, $h_2(x) = O(x^{-7/4})$ instead of (5.7). Then the system corresponding to (5.8) has the coefficient matrix

$$\begin{aligned} B(x) = \big(1 + \tfrac{3}{8}x^{-1} + O(x^{-3/2})\big)I + \lambda^{1/2}x^{-3/4}(1 + O(x^{-1/2}))J \\ + O(x^{-7/4})K - \tfrac{3}{8}x^{-1}(1 + O(x^{-1/2}))L. \end{aligned}$$

By the same argument as in section 5.2, we may find a transformation of the form $\mathbf{v}(x) = (I + O(x^{-1/4})K)\mathbf{y}(x)$ so that this is reduced to a system with the coefficient matrix

$$E(x) = \big(1 + \tfrac{3}{8}x^{-1} + O(x^{-3/2})\big)I + \lambda^{1/2}x^{-3/4}(1 + O(x^{-1/2}))J + O(x^{-7/4})K + O(x^{-3/2})L.$$

Observe that $\sigma(x) = 4\lambda^{1/2}x^{1/4}$ satisfies $\mathrm{e}^{\sigma(x+1)-\sigma(x)} = 1 + \lambda^{1/2}x^{-3/4} + O(x^{-3/2})$. Under the condition $\lambda < 0$ i.e. $\lambda^* > 0$, we have $\mathrm{e}^{\pm\sigma(x)} = O(1)$ in $D_{+\infty}^{3/4}(r_1)$. Hence the associated linear system admits the fundamental matrix solution

$$U(x) = \begin{pmatrix} 1 + O(x^{-1/4}) & 1 + O(x^{-1/4}) \\ -\lambda^{1/2}x^{-3/4}(1 + O(x^{-1/4})) & \lambda^{1/2}x^{-3/4}(1 + O(x^{-1/4})) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\sigma(x)} & 0 \\ 0 & \mathrm{e}^{\sigma(x)} \end{pmatrix} x^{3/8},$$

$$\sigma(x) = 4\lambda^{1/2}x^{1/4}$$

with the Casorati determinant $\det U(x) = 2\lambda^{1/2}(1 + O(x^{-1/4}))$ in $D_{+\infty}^{3/4}(r_2)$ for some $r_2 > 0$. Using this, we may show that the equivalent nonlinear equation has a solution $\omega(x)$ such that $\omega(x) \sim 0$ as $x \to \infty$ through $D_{+\infty}^{3/4}(r)$ for some $r > r_2$, which implies the asymptotic expression of $\varphi_-^*(x)$ in theorem 2.7. The remaining parts of theorems 2.7 and 2.8 are derived by using the substitutions $(\gamma, y(x)) \mapsto (-\gamma, -y(x))$ and $(\alpha, x, y(-x)) \mapsto (-\alpha, -x, y(x))$, respectively.

8. Proofs of the results on (1.3)

To compute the formal solution we substitute $y(x) = cx^{1/2} + y_0(x)$ into (1.3). The left-hand member is

$$y_0(x+1) + y_0(x-1) + 2cx^{1/2} - \frac{c}{4}x^{-3/2} + \cdots,$$

and the right-hand member is

$$-\frac{\gamma}{c^2}x^{-1}\sum_{j\geqslant 0}(-1)^j Y^j - \frac{\alpha}{c}x^{1/2}\left(1 + \frac{\beta}{\alpha}x^{-1}\right)\left(1 + \frac{x^{-1/2}}{c}y_0(x)\right)\sum_{j\geqslant 0}(-1)^j Y^j$$

with

$$Y := -c^{-2}x^{-1} + 2c^{-1}x^{-1/2}y_0(x) + c^{-2}x^{-1}y_0(x)^2.$$

Comparing the coefficients of $x^{1/2}$ on both sides, we have $c = c_\pm = \pm\sqrt{-\alpha/2}$. Then the equation is written in the form

$$\begin{aligned} y_0(x+1) + 2y_0(x) + y_0(x-1) = &-\frac{\beta-2}{c_\pm}x^{-1/2} + \frac{2\gamma}{\alpha}x^{-1} + O_{(-1/2)}[x^{-3/2}] \\ &+ \left(\frac{12-2\beta}{\alpha}x^{-1} - \frac{4\gamma}{\alpha c_\pm}x^{-3/2} + O_{(-1/2)}[x^{-2}]\right)y_0(x) \\ &+ \left(\frac{2}{c_\pm}x^{-1/2} + O_{(-1/2)}[x^{-3/2}]\right)y_0(x)^2 + \cdots + O_{(-1/2)}[x^{-(j-1)/2}]y_0(x)^j + \cdots. \end{aligned}$$

By $y_0(x) = -(\beta-2)x^{-1/2}/(4c_\pm) + y_1(x)$, this is changed into

$$\begin{aligned} y_1(x+1) + 2y_1(x) + y_1(x-1) = &\frac{2\gamma}{\alpha}x^{-1} + O_{(-1/2)}[x^{-3/2}] \\ &+ \left(\frac{8}{\alpha}x^{-1} - \frac{4\gamma}{\alpha c_\pm}x^{-3/2} + O_{(-1/2)}[x^{-2}]\right)y_1(x) \\ &+ \left(\frac{2}{c_\pm}x^{-1/2} + O_{(-1/2)}[x^{-3/2}]\right)y_1(x)^2 + \cdots + O_{(-1/2)}[x^{-(j-1)/2}]y_1(x)^j + \cdots, \end{aligned}$$

and by $y_1(x) = \gamma x^{-1}/(2\alpha) + y_2(x)$ we have

$$\begin{aligned} y_2(x+1) + 2y_2(x) + y_2(x-1) = &\,O_{(-1/2)}[x^{-3/2}] \\ &+ \left(\frac{8}{\alpha}x^{-1} - \frac{2\gamma}{\alpha c_\pm}x^{-3/2} + O_{(-1/2)}[x^{-2}]\right)y_2(x) + \cdots + O_{(-1/2)}[x^{-(j-1)/2}]y_2(x)^j + \cdots. \end{aligned}$$

Repeating further transformations, we obtain the formal solutions $\Psi_\pm(\alpha,\beta,\gamma,x) = \pm\sqrt{-\alpha/2}x^{1/2} + \sum_{j\geqslant 1} d_j^\pm x^{-j/2}$ in proposition 2.10. By $y(x) = -\sqrt{-\alpha/2}x^{1/2} + \xi_-(x) + z(x)$ with $\xi_-(x) \sim \sum_{j\geqslant 1} d_j^- x^{-j/2}$, equation (1.3) is changed into an equivalent equation of the form (4.1) with

$$g(x) = \lambda x^{-1} + \mu x^{-3/2} + O(x^{-2}), \qquad \lambda := -\frac{8}{\alpha}, \qquad \mu := -\frac{2\gamma}{\alpha\sqrt{-\alpha/2}},$$

more precisely, with $g(x)$ expanded into the asymptotic series

$$g(x) \sim \lambda x^{-1} + \mu x^{-3/2} + O_{(-1/2)}[x^{-2}]$$

as $x \to \infty$ through a half-plane (cf remark 4.1, and, for the symbol $O_{(-1/2)}[x^{-2}]$, see section 3). Let us start with the associated system of the form (5.3) with $g(x)$ given above. Observe that

$$\sqrt{g(x) + g(x)^2/4} \sim \lambda^{1/2}x^{-1/2}\left(1 + \frac{\mu}{2\lambda}x^{-1/2} + O_{(-1/2)}[x^{-1}]\right).$$

Then the quantities given by (5.5) also admit the asymptotic representations

$$h_1(x) \sim \frac{x^{-1}}{4}(1 + O_{(-1/2)}[x^{-1/2}]), \qquad h_2(x) \sim O_{(-1/2)}[x^{-3/2}],$$

and hence we obtain the system

$$\mathbf{v}(x+1) = B(x)\mathbf{v}(x) \tag{8.1}$$

with

$$\begin{aligned} B(x) = &\left(1 + \frac{2\lambda+1}{4}x^{-1} + O(x^{-3/2})\right)I + \left(\lambda^{1/2}x^{-1/2} + \frac{\lambda^{-1/2}\mu}{2}x^{-1} + O(x^{-3/2})\right)J \\ &+ O(x^{-3/2})K - \left(\frac{x^{-1}}{4} + O(x^{-3/2})\right)L. \end{aligned}$$

Note that each coefficient is also expanded into an asymptotic series in powers of $x^{-1/2}$ as $x \to \infty$ through a half-plane.

Let ν_0 be a given positive number. By the same argument as in section 5.2 concerning (5.10) with (5.11), we may find a transformation of the form $\mathbf{v}(x) = (I + p_1 x^{-1/2} K)\mathbf{v}_1(x)$ such that the coefficient of L is reduced to $O(x^{-3/2})$. Choosing $\mathbf{v}_1(x) = (I + p_2 x^{-1} K)\mathbf{v}_2(x)$ suitably, we may further reduce the coefficient of L to $O(x^{-2})$. Repeating this procedure we obtain the transformation $\mathbf{v}(x) = (I + O(x^{-3/2})I + O(x^{-1/2})K)\mathbf{v}_*(x)$, by which (8.1) is changed into

$$\mathbf{v}_*(x+1) = B_*(x)\mathbf{v}_*(x) \tag{8.2}$$

with

$$B_*(x) = b_0^*(x)I + b_1^*(x)J + b_2^*(x)K + b_3^*(x)L,$$
$$b_0^*(x) = 1 + \frac{2\lambda+1}{4}x^{-1} + O(x^{-3/2}),$$
$$b_1^*(x) = \lambda^{1/2}x^{-1/2} + \frac{\lambda^{-1/2}\mu}{2}x^{-1} + O(x^{-3/2}),$$
$$b_2^*(x) = O(x^{-3/2}), \qquad b_3^*(x) = O(x^{-\nu_0}),$$

where each coefficient is expanded into an asymptotic series in powers of $x^{-1/2}$. To reduce the power exponent of the coefficient of K, consider a transformation of the form $\mathbf{v}_*(x) = (I + q(x)L)\mathbf{v}_{**}(x)$. A straightforward computation leads us to the following:

$$\mathbf{v}_{**}(x+1) = B_{**}(x)\mathbf{v}_{**}(x)$$

with

$$\begin{aligned} B_{**}(x) &= (I + q(x+1)L)^{-1}B_*(x)(I + q(x)L) \\ &= \frac{1}{1-q(x+1)^2}[((1 - q(x)q(x+1))b_0^*(x) - (q(x+1) - q(x))b_3^*(x))I \\ &\quad + ((1 + q(x)q(x+1))b_1^*(x) - (q(x+1) + q(x))b_2^*(x))J \\ &\quad + ((1 + q(x)q(x+1))b_2^*(x) - (q(x+1) + q(x))b_1^*(x))K \\ &\quad + ((1 - q(x)q(x+1))b_3^*(x) - (q(x+1) - q(x))b_0^*(x))L]. \end{aligned}$$

Using this fact, we may also construct a transformation of the form

$$\mathbf{v}_*(x) = (I + O(x^{-3/2})I + O(x^{-1})L)\mathbf{y}(x)$$

such that (8.2) is taken into

$$\begin{aligned} &\mathbf{y}(x+1) = E(x)\mathbf{y}(x), \\ &E(x) = (b_0^*(x) + O(x^{-3/2}))I + (b_1^*(x) + O(x^{-3/2}))J + O(x^{-\nu_0})K + O(x^{-\nu_0})L. \end{aligned} \tag{8.3}$$

Finally putting

$$\mathbf{y}(x) = \begin{pmatrix} e^{-\sigma(x)}x^{-\lambda^{-1/2}\mu/2} & 0 \\ 0 & e^{\sigma(x)}x^{\lambda^{-1/2}\mu/2} \end{pmatrix} x^{1/4}\mathbf{w}(x), \qquad \sigma(x) = 2\lambda^{1/2}x^{1/2},$$

we have the system

$$\mathbf{w}(x+1) = Q(x)\mathbf{w}(x),$$
$$Q(x) = I + \begin{pmatrix} \varepsilon_{11}(x) & e^{2\sigma(x)}x^{\lambda^{-1/2}\mu}\varepsilon_{12}(x) \\ e^{-2\sigma(x)}x^{-\lambda^{-1/2}\mu}\varepsilon_{21}(x) & \varepsilon_{22}(x) \end{pmatrix}$$
$$\varepsilon_{11}(x) = O(x^{-3/2}), \qquad \varepsilon_{22}(x) = O(x^{-3/2}), \qquad \varepsilon_{12}(x) = O(x^{-\nu_0}), \qquad \varepsilon_{21}(x) = O(x^{-\nu_0}).$$

If $\alpha > 0$, then $\lambda^{1/2}$ is purely imaginary, and hence $e^{\pm\sigma(x)} = O(1)$ in $D^{1/2}_{+\infty}(r_1)$ $(r_1 > 0)$. If ν_0 is chosen sufficiently large, then $Q(x) - I = O(x^{-3/2})$. Thus we obtain the fundamental matrix solution

$$U(x) = \begin{pmatrix} 1 + O(x^{-1/2}) & 1 + O(x^{-1/2}) \\ -\lambda^{1/2}x^{-1/2}(1 + O(x^{-1/2})) & \lambda^{1/2}x^{-1/2}(1 + O(x^{-1/2})) \end{pmatrix} \times \begin{pmatrix} e^{-\sigma(x)}x^{-\lambda^{-1/2}\mu/2} & 0 \\ 0 & e^{\sigma(x)}x^{\lambda^{-1/2}\mu/2} \end{pmatrix} x^{1/4}.$$

Using this, we get the asymptotic solution $\psi_-(x)$ of (1.3) under the condition $\alpha > 0$ by the same argument as in section 6. The condition $\alpha > 0$ is invariant under the substitution $(\gamma, y(x)) \mapsto (-\gamma, -y(x))$, and hence equation (1.3) simultaneously possesses the asymptotic solution $\psi_+(x) = -\psi_-(\alpha, \beta, -\gamma, x)$, which implies theorem 2.11. Using the substitution $(\alpha, x, y(x)) \mapsto (-\alpha, -x, y(-x))$, we obtain theorem 2.12.

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